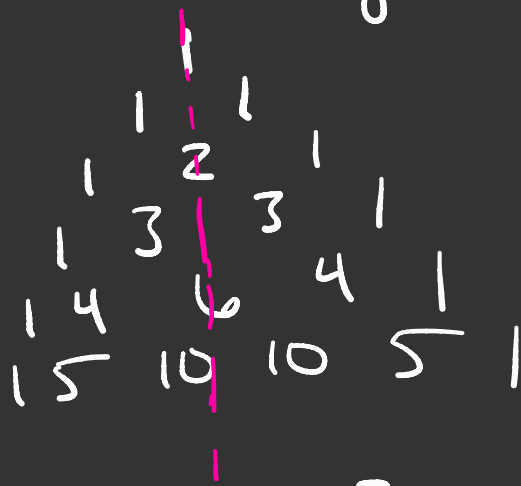


Challenge Problem: Pascal's Triangle is symmetric



$$C(n, k) = C(n, n-k)$$

Why does $C(10, 3) = C(10, 7)$? $A = \{a, b, c, \dots, j\}$

Choosing a 3-element subset is the same as choosing the 7 elements not in the subset.

Motivating Example: 21 students in class. How many ways can we split into teams so that
6 students on red team
7 students on blue team
8 students on green team

$$\begin{aligned} \# \text{ ways to split into teams} &= \binom{\# \text{ ways to choose red team}}{\binom{\# \text{ ways to choose blue team}}{\binom{\# \text{ ways to choose green team}}} \\ &= C(21, 6) \times C(15, 7) \times C(8, 8) \\ &= \frac{21!}{6! 15!} \times \frac{15!}{7! 8!} \times \frac{8!}{8! 0!} = \frac{21!}{6! 7! 8!} \end{aligned}$$

Definition: A set S is partitioned into k nonempty set

A_1, A_2, \dots, A_k if

• $A_i \cap A_j = \emptyset$ when $i \neq j$ (all disjoint)

• $A_1 \cup A_2 \cup \dots \cup A_k = S$

Example: $S =$ students in class

$A_1 =$ red team $A_2 =$ blue team $A_3 =$ green team

In general, a set with n elements can be partitioned into k ordered subsets of r_1, r_2, \dots, r_k elements ($r_1 + r_2 + \dots + r_k = n$) in

Partitions can be told apart (team colors, committee names, etc.)

$$\binom{n}{r_1, r_2, \dots, r_k} = \frac{n!}{r_1! r_2! \dots r_k!} \text{ ways}$$

Example: 15 players on basketball team. How many ways are there to choose 1st, 2nd, 3rd string teams, each with 5 players?

$$\binom{15}{5, 5, 5} = \frac{15!}{5! 5! 5!} = 756,756 \text{ ways}$$

Special case: Partition into 2 sets

$$\binom{15}{7,8} = \frac{15!}{7!8!} = C(15,7) \text{ or } C(15,8)$$

$$\binom{n}{r_1, r_2} = C(n, r_1) = C(n, r_2)$$
$$(r_1 + r_2 = n)$$

What about unordered partitions?

Example: 21 students in class. How many ways to split into 3 groups of 7?

i) Start with ordered partitions $\binom{21}{7,7,7} = \frac{21!}{7!7!7!}$

ii) How much did we overcount by?

R G B

G R B

R B G

G B R

B R G

B G R

overcounted by
factor of $3! = 6$

$$\frac{\binom{21}{7,7,7}}{3!} = \frac{21!}{3!7!7!7!}$$

ways

In general, a set of n elements can be partitioned into k unordered subsets of r elements each ($kr=n$)

is

$$\frac{1}{k!} \binom{n}{\underbrace{r, r, \dots, r}_{k \text{ copies}}} = \frac{n!}{k! (r!)^k}$$

Example: 21 students. How many ways to split into groups of size 5, 5, 5, 6?

i) $\binom{21}{5,5,5,6} = \frac{21!}{5!5!5!6!}$ ordered partitions

ii) Overcounted by $\underbrace{\quad}_{5 \text{ students}}$
 R, G, B , P has to have 6 students in it

can only switch around these groups

overcounted by 3!

$$\# \text{ unordered partitions} = \frac{1}{3!} \binom{21}{5,5,5,6}$$

Example: 21 students. How many ways to split into groups of 2, 2, 2, 2, 3, 3, 7?

$$\frac{\binom{21}{2, 2, 2, 2, 3, 3, 7}}{4! \cdot 2!} = \frac{21!}{4!(2!)^4 2!(3!)^2 7!}$$

4 groups of 2

2 groups of 3