Challenge Problem: Pascals Triangle is symmetric

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$C(n,k) = C(n,n-k)$$

Why does $C(10,3) = C(10,7)$? $A = \{a,b,c,\ldots,j\}$

Choosing a 3-element subset is the same as choosing the 7 elements <u>not</u> in the subset.

Motivating Example: 21 students in class. How many ways can we split into teams so that 6 students on red team
7 students on blue team
8 students on green team

$$\left(\begin{array}{c}\#\text{ ways to}\\ \text{split into}\\ \text{teams}\end{array}\right) = \left(\begin{array}{c}\#\text{ ways to}\\ \text{choose red}\\ \text{team}\end{array}\right) \times \left(\begin{array}{c}\#\text{ ways to}\\ \text{choose blue}\\ \text{team}\end{array}\right) \times \left(\begin{array}{c}\#\text{ ways to}\\ \text{choose green}\\ \text{team}\end{array}\right)$$

$$= C(21,6) \times C(15,7) \times C(8,8)$$

$$= \frac{21!}{6!\,15!} \times \frac{15!}{7!\,8!} \times \frac{8!}{8!\,0!} = \frac{21!}{6!\,7!\,8!}$$

Definition: A set S is **partitioned** into k nonempty set $A_1, A_2, \ldots, A_k$ if

- $A_i \cap A_j = \emptyset$ when $i \neq j$ (all disjoint)
- $A_1 \cup A_2 \cup \cdots \cup A_k = S$

Example: S = students in class

A_1 = red team A_2 = blue team A_3 = green team

In general, a set with n elements can be partitioned into k <u>ordered</u> subsets of $r_1, r_2, \ldots, r_k$ elements $(r_1 + r_2 + \cdots + r_k = n)$ in

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \cdots r_k!} \quad \text{ways}$$

Portions can be told apart (team colors, committee names, etc.)

Example: 15 players on basketball team. How many ways are there to choose 1^{st}, 2^{nd}, 3^{rd} string teams, each with 5 players?

$$\binom{15}{5,5,5} = \frac{15!}{5!5!5!} = 756{,}756 \text{ ways}$$

Special case: Partition into 2 sets

$$\binom{15}{7,8} = \frac{15!}{7!\,8!} = C(15,7) \text{ or } C(15,8)$$

$$\binom{n}{r_1, r_2} = C(n, r_1) = C(n, r_2)$$

$$(r_1 + r_2 = n)$$

What about unordered partitions?

Example: 21 students in class. How many ways to split into 3 groups of 7?

i) Start with ordered partitions $\binom{21}{7,7,7} = \frac{21!}{7!7!7!}$

ii) How much did we overcount by?

$$\left.\begin{matrix} R\,G\,B \\ G\,R\,B \\ R\,B\,G \\ G\,B\,R \\ B\,R\,G \\ B\,G\,R \end{matrix}\right\}$$ overcounted by factor of $3! = 6$

$$\boxed{\frac{\binom{21}{7,7,7}}{3!} = \frac{21!}{3!7!7!7!} \text{ ways}}$$

In general, a set of n elements can be partitioned into k <u>unordered</u> subsets of r elements each $(kr = n)$ is

$$\frac{1}{k!}\binom{n}{\underbrace{r, r, \ldots, r}_{k \text{ copies}}} = \frac{n!}{k!\,(r!)^k}$$

Example: 21 students. How many ways to split into groups of size 5, 5, 5, 6?

i) $\binom{21}{5,5,5,6} = \frac{21!}{5!5!5!6!}$ <u>ordered</u> partitions

ii) Overcounted by

$$\overbrace{R,\ G,\ B}^{5 \text{ students}}_{\underbrace{\hphantom{R,\ G,\ B}}_{\text{can only switch around these groups}}},\quad P$$

P ← has to have 6 students in it

can only switch around these groups ⟶ overcounted by 3!

$$\#\text{unordered partitions} = \frac{1}{3!}\binom{21}{5,5,5,6}$$

Example: 21 students. How many ways to split into groups of 2, 2, 2, 2, 3, 3, 7?

$$\frac{\binom{21}{2,2,2,2,3,3,7}}{\underset{\text{4 groups of 2}}{4!} \cdot \underset{\text{2 groups of 3}}{2!}} = \frac{21!}{4!(2!)^4 2!(3!)^2 7!}$$