Combinations

Challenge Problems: What value of k will make $P(11,k)$ the biggest/smallest?

Biggest: $P(11,11) = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdots 2 \cdot 1$

$P(11,10)$ =

Smallest: $P(11,2) = 11 \cdot 10$

$P(11,1) = 11$

$P(11,0) = 1$

$P(n,k) = n(n-1)(n-2)\cdots$

$P(11,23) = 11 \cdot 10 \cdots 0 \cdot (-1)(-2) \cdots = 0$

Combinations: Subset of given set without regard to arrangement

Same as permutations but <u>order doesn't matter</u>

Natation: $C(n,k)$ = "combinations of n things taken k at a time"

= "n choose k"

= # ways to choose k elements from an n-elements set

Example: $C(3,2) = 3$

$\{a,b,c\} \xrightarrow{\text{2-element subsets}}$ $\{a,b\}$, $\{a,c\}$, $\{b,c\}$

Question: Which is bigger: $\boxed{P(n,k)}$ or $C(n,k)$?

Different perms $\rightarrow \{a,b\}$, $\{b,a\}$ $\rightarrow \{a,b\}$ Same comb

Formula for $C(n,k)$:

Example: $C(10,3)$

i) Count all permutation of 10 things taken 3 at a time

$$P(10,3) = \frac{10!}{7!}$$

ii) How much did we overcount by?

$$\{A, B, C, D, E, \ldots, J\}$$

$$\left.\begin{array}{l} A, B, C \\ A, C, B \\ B, A, C \\ B, C, A \\ C, A, B \\ C, B, A \end{array}\right\} \longrightarrow \{A, B, C\}$$

Overcounted by factor of $6 = 3!$

$$\leadsto \underline{C(10, 3)} = \frac{P(10, 3)}{3!} = \frac{10 \cdot \cancelto{3}{9} \cdot \cancelto{4}{8}}{\cancel{3} \cdot \cancel{2} \cdot 1} = 120$$

$$\leadsto \text{In general, } C(n, k) = \frac{P(n, k)}{k!} = \frac{\frac{n!}{(n-k)!}}{k!} = \boxed{\frac{n!}{k!\,(n-k)!}}$$

Special Cases: $$C(n,1) = \frac{n!}{1!\,(n-1)!} = \frac{n!}{(n-1)!} = \frac{n\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{(2)}\cancel{(1)}}{\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{(2)}\cancel{(1)}}$$

$$= n$$

$$C(n,0) = \frac{n!}{0!\,n!} = 1$$

$$C(n,n) = 1 = \frac{n!}{n!\,0!}$$

Example: Making a Tik Tok dance. All dances use same 8 dance moves

a) How many ways to choose 4 dance moves to put in dance? (Not ordering them yet)

$$\frac{8!}{4!(8-4)!} = C(8,4) = 60$$

b) How many ways to choose and order 4 moves into a dance?

$$\boxed{P(8,4) = \frac{8!}{4!} = 1680}$$

Example: Lunch time

$$\left.\begin{array}{l}\bullet\ 4\text{ meat}\\ \bullet\ 6\text{ veggies}\\ \bullet\ 5\text{ desserts}\end{array}\right\}\text{Choices}$$

$$\underset{\text{meats}}{A_1} \to \underset{\text{veggies}}{A_2} \to \underset{\text{desserts}}{A_3}$$

How many can make a meal of 2 meats, 2 veggies, 4 desserts?

$$\#\text{ such meals} = (\#\text{ ways to choose meats}) \times (\#\text{ ways to choose veggies}) \times (\#\text{ ways to choose desserts})$$

$$= C(4,2) \times C(6,2) \times C(5,4)$$

$$= 450$$

Binomial Theorem:

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Pattern?

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

$$x^3y$$

How many x^3y's show up in the expansion?

$C(4, 1)$ ← Pick one to contribute y

↖ 4 terms to choose from

The coefficient of x^3y in expansion is $C(4,1)$

$$(x+y)^4 = \underset{\substack{\uparrow \\ C(4,0)}}{\underline{1}}\,x^4 + \underset{\substack{\uparrow \\ C(4,1)}}{\underline{4}}\,x^3y + \underset{\substack{\uparrow \\ C(4,2)}}{\underline{6}}\,x^2y^2 + \underset{\substack{\uparrow \\ C(4,3)}}{\underline{4}}\,xy^3 + \underset{C(4,4)}{\underline{1}}\,y^4$$

In general,

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \text{---} + C(n,n-1)xy^{n-1} + C(n,n)y^n$$