

# Combinations

Challenge Problems: What value of  $k$  will make  $P(11, k)$  the biggest/smallest?

Biggest:  $P(11, 11) = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \dots 2 \cdot 1$   
 $P(11, 10) \approx$

Smallest:  $P(11, 2) = 11 \cdot 10$

$$P(11, 1) = 11$$

$$P(11, 0) = 1$$

$$P(n, k) = n(n-1)(n-2) \dots$$

$$P(11, 23) = 11 \cdot 10 \dots 0 \cdot (-1) \cdot (-2) \dots = 0$$

Combinations: Subset of given set without regard to arrangement

Same as permutations but order doesn't matter

Notation:  $C(n, k)$  = "combinations of  $n$  things taken  $k$  at a time"

= "n choose k"

= # ways to choose  $k$  elements from an  $n$ -elements set

Example:  $C(3, 2)$   
"3"

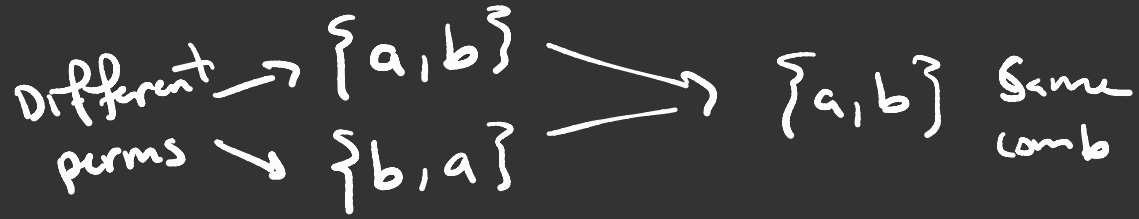
$\{a, b, c\}$  <sup>2-element</sup>  
subsets

$\{a, b\}$

$\{a, c\}$

$\{b, c\}$

Question: Which is bigger:  $P(n,k)$  or  $C(n,k)$ ?



Formula for  $C(n,k)$ :

Example:  $C(10,3)$

i) Count all permutation of 10 things taken 3 at a time

$$P(10,3) = \frac{10!}{7!}$$

ii) How much did we overcount by?

$\{A, B, C, D, E, \dots, J\}$

A, B, C

A, C, B

B, A, C

B, C, A

C, A, B

C, B, A

$\{A, B, C\}$

Overcounted by  
factor of  $6 = 3!$

$$\rightarrow \underline{C(10, 3)} = \frac{P(10, 3)}{3!} = \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}}}{3 \cdot 2 \cdot 1} = 120$$

$$\rightarrow \text{In general, } C(n, k) = \frac{P(n, k)}{k!} = \frac{\frac{n!}{(n-k)!}}{k!} = \boxed{\frac{n!}{k! \cdot (n-k)!}}$$

Special Cases:  $C(n, 1) = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\dots(2)(1)}{(n-1)(n-2)\dots(2)(1)}$

$$= \frac{n}{1}$$

$$C(n, 0) = \frac{n!}{0!n!} = 1$$

$$C(n, n) = 1 = \frac{n!}{n!0!}$$

Example: Making a Tik Tok dance. All dances use  
same 8 dance moves

a) How many ways to choose 4 dance moves to put in  
dance? (Not ordering them yet)

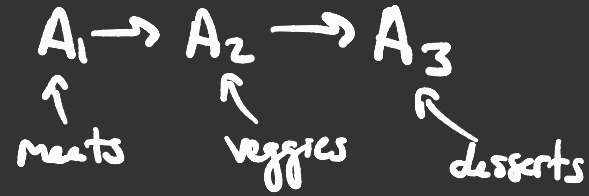
$$\frac{8!}{4!(8-4)!} = \underline{\underline{C(8,4)}} = 60$$

b) How many ways to choose and order 4 moves into  
a dance?

$$\boxed{P(8,4) = \frac{8!}{4!} = 1680}$$

Example: Lunch time

- 4 meats
  - 6 veggies
  - 5 desserts
- } Choices



How many can make a meal of 2meats, 2veggies,  
4 desserts?

$$\# \text{ such meals} = (\# \text{ ways to choose meats}) \times (\# \text{ ways to choose veggies}) \\ \times (\# \text{ ways to choose desserts})$$

$$= C(4, 2) \times C(6, 2) \times C(5, 4)$$

$$= 450$$

# Binomial Theorem:

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = \underline{x^4} + \underline{4x^3y} + \underline{6x^2y^2} + \underline{4xy^3} + \underline{y^4}$$

Pattern?

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

$x^3y$

How many  $x^3y$ 's show up in the expansion?

$C(4, 1)$  Pick one to contribute  $y$   
4 terms to choose from

The coefficient of  $x^3y$  in expansion is  $C(4, 1)$



$$(x+y)^4 = \frac{1}{\uparrow} x^4 + \frac{4}{\uparrow} x^3 y + \frac{6}{\uparrow} x^2 y^2 + \frac{4}{\uparrow} x y^3 + \frac{1}{\uparrow} y^4$$

$C(4,0) \quad C(4,1) \quad C(4,2) \quad C(4,3) \quad C(4,4)$

In general,

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \\ \dots + C(n,n-1)xy^{n-1} + C(n,n)y^n$$