On Wednesday, started talking about matrices

Talked about adding/multiplying matrices

Example:

Match ✓

$$\overset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}} = \overset{2\times 1}{\begin{bmatrix} 1\times(-1) + 2\times 1 + 3\times 2 \\ 3\times(-1) + 2\times 1 + 1\times 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Special 2×2 matrices: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Say M is any 2×2 matrix

- $M + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = M$

- $M \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- $M \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} a\times 1 + b\times 0 & a\times 0 + b\times 1 \\ c\times 1 + d\times 0 & c\times 0 + d\times 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= M$$

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot M = M$

Application: Can represent systems of equations as matrix equations.

$$\begin{cases} 2x + 3y = 10 \\ 3x + 4y = 11 \end{cases}$$

Notice

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ 3x+4y \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

$$\begin{matrix} 2x+3y=10 \\ 3x+4y=11 \end{matrix} \quad \xleftrightarrow{\text{Equivalent Statements}} \quad \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

This is powerful because some matrices have an <u>inverse</u> (B is an inverse of A if $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

For example,

$$\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 \times 2 + 3 \times 3 & -4 \times 3 + 3 \times 4 \\ 3 \times 2 + (-2) \times 3 & 3 \times 3 + (-2) \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, if we want to solve

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

Multiply both sides on the left by $\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$:

$$\underbrace{\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

$$\boxed{\begin{aligned} 2x + 3y &= 10 \\ 3x + 4y &= 11 \end{aligned}}$$

$$= \begin{bmatrix} -4 \times 10 + 3 \times 11 \\ 3 \times 10 + (-2) \times 11 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 8 \end{bmatrix}$$

Check:

$$2(-7) + 3(8) = 10 \checkmark$$

$$3(-7) + 4(8) = 11 \checkmark$$

Hard question: Which 2×2 matrices have inverses?

Example: Does $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have an inverse?

Let's say $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is its inverse.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+2b & 2a+4b \\ c+2d & 2c+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Looking at top row,

$$a + 2b = 1 \rightsquigarrow a = 1 - 2b$$

$$2a + 4b = 0$$

$$2(1-2b) + 4b = 0$$

$$2 - 4b + 4b = 0$$

$$2 = 0$$

No solution

Problem: Second row of matrix is multiple of first row.

If this does not happen, then the 2×2 matrix has an inverse.